A260 Final Exam
Group Problems Due Tuesday, May 3
Solo Problems Due Friday, May 6

Instructions

- Do the group problems first. Do not open or look at the solo problems until you have completed the group problems.
- Group problems must be done in a group of two or three people. Do not discuss them with anybody outside of your group.
- One set of solutions should be turned in for each group. Names of all members of the group must be on the solution set. **Keep both the problem statements and a *copy* of your solutions for the group problems, as you will need them for the solo problems.**
- Once you have turned in the group problems, you may do the solo problems. **Do not discuss the solo problems with anybody before Saturday, May 7.**
- You may use Hartle, your class notes, and any materials (including problem sets and solutions) that I have posted to the course web page. You may use things like *Mathematica*, although I reserve the right to make fun of you. You may numerically integrate if it is *really* necessary.
- There is no time limit, but please do not kill yourself.
- You may consult me at any time.
- Do not fold, mutilate, or spindle.
- If taken internally, contact a physician immediately.

In non-geometrized units, we have:

$$
\begin{aligned}
G &= 6.67 \times 10^{-11}\,\frac{\mathrm{m}^3}{\mathrm{kg\ s}^2} \\
c &= 3.00 \times 10^{8}\,\frac{\mathrm{m}}{\mathrm{s}} \\
H_0 &= \left(\frac{\dot{a}}{a}\right)_{t=t_0} = 71\,\frac{\mathrm{km\ s}^{-1}}{\mathrm{Mpc}} = \frac{1}{1.4 \times 10^{10}\,\mathrm{yr}} \\
M_\odot &= 2 \times 10^{30}\,\mathrm{kg}
\end{aligned}
$$

Group Problem #1. You are bold. You are very bold. You wish to hover just over the event horizon of a black hole. You are so bold that you have received the unlimited government funding necessary to produce rockets with enough thrust to hover you over the event horizon of a black hole for a while, and to escape afterwards. Unfortunately, there is one condition on the funding. You must survive the trip. Work, work, work.

This means you need to choose your black hole wisely. See, the curvature of space near the event horizons of some black holes is so huge that the tidal forces (difference in acceleration from one side of an object to another) can tear a normal person apart. Some black holes are safer than others, however. In this problem, you will figure out how to choose your black hole.

Consider only Schwarzschild black holes. It will be of use to you to know that on page 546 of Hartle, you can find the metric and all of the Christoffel Symbols for this geometry.

For this problem, define the following variables. h is the coordinate distance you will hover above the event horizon. We will assume that $h \ll 2M$, because you want to get close. We'll assume you're hovering head-up and feet-down. Δr will be the difference between the r-coordinate of your feet and of your head. l is your height. M is the mass of the black hole. Until we get numerical, express your answer in terms of these quantities, except where noted.

(a) When you are hovering, what are the components (in Schwarzschild coordinates) of your 4-velocity u^α?

(b) Were you fall freely, you would move according to the geodesic equation:

$$\frac{du^\alpha}{d\tau} = -\Gamma^\alpha_{\beta\gamma}\, u^\beta\, u^\gamma$$

Calculate the components of your 4-acceleration a^α, again in Schwarzschild coordinates, were you to fall freely.

(c) Consider the r-component of your 4-acceleration. What is the *difference* in this component between your feet and your head? Express your answer in terms of M, h, and Δr. (Use the binomial theorem to simply this expression, knowing that $\Delta r \ll 2M$.)

(d) Of course, Δr isn't terribly useful, since it's just a couple of coordinates and we don't really know what it is at the moment. Write down an expression for Δr in terms of l and the other quantities of this problem.

(e) You're bold, but you can only take so much tearing. You want the difference in the accelerations between your feet and your head to be $< 10g$, that is, less than $100\,\mathrm{m\,s^{-2}}$. (Make sure you properly convert this to geometrized units!) So that you've got a little room to wiggle about, fix $h = 10$m. You are very tall, so use $l = 2m$ for your height.

What are the masses of the black holes you can safely hover over? Express your answer in terms of solar masses, knowing that (in geometrized units) $1\,M_\odot = 1480\,\mathrm{m}$. Where might you find such a black hole?

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Group Problem #2. Pick a comfortable black hole from the previous problem— one where the tidal acceleration between your foot and your head is just $1g$. Go hover over it.

Are you back already? You can't be, because there's time dilation. So calculate that instead of really going. (This isn't a lab class.)

(a) If you hover at $h = 10$m of coordinate distance over the event horizon of the black hole for 1 year (as measured on your watch) (assuming your watch is at 10m above the horizon, and ignoring any complications that come from your head and your feet aging at different rates), how much time passes in the outside world?

(b) Suppose that you want to achieve the same feat of time dilation, but rather through special relativity. Assume you're off in flat space and moving at constant velocity. How fast would you need to go in order to have the same amount of time pass in the outside world as you found in part (a) while one year elapses for you?

(c) Consider again just special relativity, in flat space. Now suppose that you go on a trip to some other star. You accelerate at a constant acceleration of $1g$ for the first half of the trip, and decelerate at a constant acceleration of $1g$ for the second half of the trip (leaving you at rest with respect to the frame you left). If you measure one year on your watch during this trip, how much time elapses in the Galaxy? (Assume you started at rest with respect to the Galaxy.)

(d) How far did you go (as measured in the frame of reference of the Galaxy) in part (c)?

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Group Problem #3. Consider the Friedmann Equation for the scale factor of the Flat Robertson-Walker metric:

$$\dot{a}^2 \; - \; \frac{8\pi\rho}{3}\,a^2 \; = \; 0$$

Recall also that we derived in class that if the Universe is filled with a perfect fluid with equation of state parameter w (where w is defined as p/ρ, p being the pressure of the fluid and ρ being the energy density), that the density varies with the scale factor according to:

$$\rho \; = \; \rho_0 \left(\frac{a}{a_0}\right)^{-3(1+w)}$$

where ρ_0 is the present density of the fluid, and a_0 is the present scale factor (usually taken to be $a_0 = 1$).

(a) Show that for a flat Universe filled only with vacuum energy ($w = -1$), the scale factor increases exponentially, i.e.

$$a \; = \; a_0\, e^{k(t-t_0)}$$

where t_0 is right now. What is k in terms of H_0, today's Hubble Constant? (This is what would be the future evolution of the Universe if the matter were all suddenly to go away, and the vacuum energy density were suddenly to jump by $\sim$50% to the critical density.)

(b) For arbitrary w, derive an expression for $\ddot{a}$, the acceleration of the scale factor.

(c) "Phantom Dark Energy" models have $w < -1$. (I think. Ask Bob Scherrer if I got my terminology right.) Consider a specific such model with $w = -2$. Derive an expression for $a(t)$. Evaluate any constants of integration you get in terms of H_0, and put in the present day density of the Universe (assuming that the full critical density is made up with this phantom dark energy). Your final expression should be for $a(t)$ in terms only of numbers, H_0, and $(t - t_0)$ (where t_0 is the age of the Universe today).

(d) For the solution in part (c), there is a time at which $\dot{a} \to \infty$. If we've got phantom dark energy with $w = -2$, when will this happen? Use our best measured value of H_0 to numerically evaluate the number of years we've got left. This is what is referred to as "Cosmic Doomsday." You have been warned. (Current data already rules out $w = -2$, but it still allows for $w < -1 \ldots$)

(e) Assume that we have phantom dark energy of some sort. Draw a plot of $a(t)$ vs. t, starting from the Big Bang and continuing through Cosmic Doomsday. Discuss how this might present a solution to the $\Omega_M - \Omega_{\text{vac}}$ coincidence problem.

. . . end of group problems.

Solo Problems

Do not look at the following pages until you have turned in the group problems.

Useful Trig Identities:

$$\cos\left(\frac{\pi}{2} - \alpha\right) = \sin(\alpha)$$

$$\sin\left(\frac{\pi}{2} - \alpha\right) = \cos(\alpha)$$

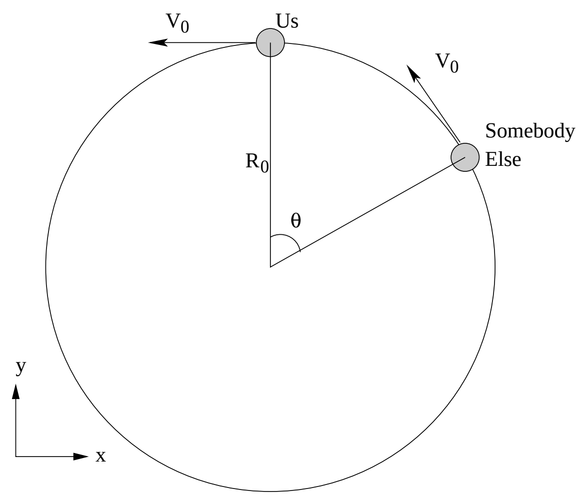
$$\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha$$

$$\sin(2\alpha) = 2\sin\alpha\cos\alpha$$

$$\cos^2\alpha + \sin^2\alpha = 1$$

The three interior angles of a triangle add up to π in flat space.

Solo Problem #1. Knowing that our Galaxy is differentially rotating (the *angular* velocity of something orbiting about the center of the Galaxy is a function of radius from the center of the galaxy), astronomers can use measured Doppler shifts to figure out where something is in the Galaxy. For this problem, you'll consider the simple case of another star which is the same distance from the center of the Galaxy as us, as shown below. (If you want to do more, take Astro 253.)



(a) Write down expressions for the 4-velocities u^α for both us and the other star. (Make sure you are clear about which is which!) Use standard Cartesian coordinates (t, x, y, z) in the frame of reference of the Galaxy. (Assume we're in flat space here.)

(b) Draw a diagram which shows the path followed by a photon emitted by the other star and detected by us. Assume that $V_0 \ll c$, so that we will not have appreciably moved from our initial position as the photon travels. (You don't have to assume this, but doing so makes the the algebra and trigonometry of this problem much less messy.)

(d) Write down the momentum 4-vector p of the photon in terms of ϕ. (Because we're in flat space, it will have the same p everywhere.) The components should be in terms of nothing other than V_0, R_0, $\theta/2$, and ω, where ω is the frequency of the photon as measured by the emitting star (labelled "Somebody Else" on the diagram). You may wish to make use of the useful trig identities on the previous page. . . .

(e) What is the frequency ω' we measure when the photon reaches us, as a fraction of ω?

(f) Does your answer to (d) make sense? In particular, rather than trying to add and subtract vectors in your head, ask yourself the following to questions. First, is the other star getting closer to us, getting farther from us, or staying the same distance away from us? Given that, what should we expect to measure for the line-of-sight velocity of the other star?

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Solo Problem #2. In this problem, you will attempt to use spacetime diagrams to demonstrate how galaxies with which we are currently in causal contact will go *out* of causal contact if the expansion of the Universe is accelerating. You will want to use the Friedmann equation from Group Problem #3, and will want to consider the Flat Robertson-Walker metric:

$$ds^2 \;=\; -dt^2 \;+\; a^2(t)\,(dx^2 + dy^2 + dz^2)$$

with $a(t_0) = 1$. (t_0 is right now.)

(a) Draw a spacetime diagram where the vertical axis is t and the horizontal axis is the *physical* distance x' from us to other galaxies which are off in the x-direction. Put us at $t = t_0, x' = 0$. Assume that $\dot{a} > 0$, but for now don't worry about $\ddot{a}$. Draw our world line (well, OK, that's the t-axis, I'll give that away), and the world line for another galaxy which is at co-moving coordinates $x = x_g$, $y = 0$, and $z = 0$.

(b) Draw a second spacetime diagram where the vertical axis is t and the horizontal axis is the comoving coordinate x. Draw the worldlines for us and for the galaxy at $x = x_g$.

(c) Draw our future lightcones on your diagram from (b) assuming that we live in a vacuum-energy dominated universe ($\rho_{\text{vac}} = \rho_{\text{critical}}$). ($w = -1$ for vacuum energy.) What is the largest co-moving coordinate for a galaxy which is in our future lightcone?

Hint: remember that a light cone is the path that a photon will follow, and that light always follows a path such that $ds^2 = 0$. You will want to refer to your answer of part (a) of Group Problem #3. Another Hint: figure out an expression $x(t)$ for light rays that cross the event of $t = t_0, x = 0$ (which is us right now).

(d) Draw our *past* lightcones on your diagram from (b), making the same assumptions as you did in (c).

(e) Now consider the event that is us at some time $t = t_1$ in the future. What is the largest co-moving coordinate that is now in our future lightcone? Is this larger or smaller than the value from (c)? Draw this point and these future lightcones on your diagram from (b). Show that you can pick an x_g such that a distant galaxy is in our future lightcone now ($t = t_0$), but is *no longer* in our future lightcone in the future (at $t = t_1$).

That galaxy used to be in causal contact with us (it could get signals at c from us), but is no longer. (Sad.)

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Solo Problem #3. This is a somewhat modified version of Hartle problem 12.13. You do not need to do anything algebraically quantitative in this problem, although if you really want to go nuts you may indulge yourself. This problem can be answered entirely by drawing and thinking about diagrams. . . as long as you properly draw the right thing. You should use and draw either Eddington-Finkelstein diagrams (similar to Figure 12.2) or a Kruskal diagrams (similar to figure 12.6) to analyze this problem. Or both, if you're ambitious.

As you perform this problem, remember that r isn't "distance" and t isn't "time" outside a Schwarzschild black hole, except for distant observers. They are coordinates that label places in spacetime. Try to think physically when interpreting something as a distance or a time.

(a) An observer falls feet first into a Schwarzschild black hole looking down at her feet. Can she see her feet when her head is crossing the horizon? If so, what radius r does she see them at? Will this look particularly strange to her? (Assume that she's done the group problems and had the foresight to select a black hole whose tidal forces wouldn't kill her before she made it all the way in.)

(b) Does she see her feet hit the singularity at $r = 0$, assuming she remains intact until her head reaches that radius?

(c) *Is it dark inside a black hole?* An observer sees a star collapsing to a black hole become dark. But would it be dark inside a black hole assuming a collapsing star continues to radiate at a steady rate as measured by an observer on its surface?