A260 Group Problems
2005 January 21

1. *[Hartle problem 2.5]* Calculate the area of a cricle of radius r (distance from center to circumference) in the two-dimensional geometry that is a surface of a sphere of radius a. Show that this reduces to $\pi\, r^2$ when $r \ll a$.

   In general, is $A < \pi r^2$, or is $A > \pi r^2$, or does the direction of the inequality depend on r? Answer this from your algebreic answer, and then answer it from just thinking about the geometry. Do you get the same answer?

2. (a) Show that the line element for the two-dimensional geometry that is a surface of a sphere can also be written as:

   $$ds^2 \;=\; a^2\,\frac{d\xi^2}{1-\xi^2} \;+\; a^2\,\xi^2\,d\phi^2$$

   with an appropriate variable substitution.

   (b) Consider the circle described by $\xi = \xi_0$. Use the line element above to calculate R, the physical radius (within this two-dimensional space) of the circle, and C, the physical circumference of this circle.

   (c) From the ratio of C/R that you get in part (b), is say that $C > 2\pi, R$ or $C < 2\pi, R$... or does it depend on the value of R? Does your answer make sense if you think about the geometry from the point of view of a hyperdimensional 3d creature such as yourself?

3. Consider the two-dimensional geometry described by the line element:

   $$ds^2 \;=\; a^2\,\frac{d\xi^2}{1+\xi^2} \;+\; a^2\,\xi^2\,d\phi^2.$$

   (So far as I know, it is not possible to represent this geometry as a surface in three-dimensional space, as we did with the geometry in problem 2. This line element, however, does describe a reasonable curved two-dimensional geometry.)

   (a) Cacluate C/R (in terms of R/a) for a circle in this geometry.

   (b) Does this geometry have $C \simeq 2\pi R$ for $R \ll a$?

   (c) Is $C > 2\pi R$, $C < 2\pi R$, or does the direction of the inequality depend on R?

4. *[Hartle problem 2.7]* Consider the following coordinate transformation from familiar rectangular coordinates (x, y), labelling points in the plane to a new set of coordinates (μ, ν):

   $$x \;=\; \mu\nu, \qquad y \;=\; \frac{1}{2}\left(\mu^2 \,-\, \nu^2\right)$$

   (a) Sketch the curves of constant μ and constant ν in the $x\,y$ plane.

   (b) Transform the line element $ds^2 \;=\; dx^2 \,+\, dy^2$ into (μ, ν) coordinates.

   (c) Do the curves of constant μ and constant ν intersect at right angles?

   (d) Find the equation of a circle of radius r centered at the origin in terms of μ and ν.

   (e) Calculate the ratio of the circumference to the diameter of a circle using (μ, ν) coordinates. Do you get the correct answer?