Astro 260
Group Problems #4: "Index Gymnastics"
2005-February-10

1. For this problem, we'll be considering 3-vectors in a normal Euclidean space. (I.e. vectors as you know and love them before talking about the 4-vectors of Special Relativity.)

   The Levi-Civita tensor ϵ_{ijk} is defined such that it is 0 if $i = j$, $i = k$, or $j = k$; it is +1 if ijk is an even permutation of 123, and it is -1 if ijk is an odd permutation of 123. Some example non-zero elements of this tensor are:

   $$\begin{aligned} \epsilon_{123} &= 1 \\ \epsilon_{213} &= -1 \\ \epsilon_{312} &= 1 \\ \epsilon_{132} &= -1 \end{aligned}$$

   (a) Consider two vectors A^i and B^j. What sort of object is $\epsilon_{ijk}A^iB^j$? (I.e. scalar, vector, rank-2 tensor, or something else?)

   (b) Write out all of the components of $\epsilon_{ijk}A^iB^j$. (Again, remember that since we're using Latin letters, they only go from 1 to 3. The implied sums are over the range 1–3.)

   (c) Look carefully at your answer to (b). Do you recognize this as a vector operation between $\vec{A}$ and $\vec{B}$ that you are familiar with?

2. OK, back to 4-vectors in SR ($g_{\alpha\beta} = \eta_{\alpha\beta}$).

   The Faraday Tensor is the rank-2 tensor that describes the electromagnetic field:

   $$F^\alpha{}_\beta = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{bmatrix}$$

   NOTE: The signs in $F^\alpha{}_\beta$ above are all backwards. (α gives the row number, β gives the column number, both starting from 0.) This was because I took the original $F^\alpha{}_\beta$ out of a book (Jackson's E&M) that uses the opposite sign convention for the line element. The solutions I will post use this form, but be aware that the definition above is wrong. See class notes from 2005/02/14.

   (a) Consider the reference frame of observer $\mathcal{O}$. In this frame, there is no magnetic field, and there is a uniform electric field in the $+\mathbf{e}_x$ direction with magnitude E_0. Write down the non-zero components of $F^\alpha{}_\beta$ in this reference frame.

   (b) Take a deep breath.

. . . continued on reverse. . .

(c) Consider another observer $\mathcal{O}'$ who is moving at velocity v in the $+x$ direction with respect to $\mathcal{O}$. Write down the components of $F^{\alpha}{}_{\beta}$ in this other frame.

3. The electromagnetic 4-force on a charge q is equal to

$$f^{\alpha} \;=\; q\, F^{\alpha}{}_{\beta}\, u^{\beta}$$

where u^{β} is the 4-velocity as usual. Consider the same to reference frames (observers $\mathcal{O}$ and $\mathcal{O}'$) and the same electromagnetic fields from the previous problem.

Consider a particle with charge q and mass m at rest in the frame of observer $\mathcal{O}$.

(a) What are the components f^{α} measured by $\mathcal{O}$?

(b) Show that f^{α} is perpendicular to the 4-velocity, i.e. $\mathbf{f} \cdot \mathbf{u} = 0$, or $\mathrm{g}_{\alpha\beta} f^{\alpha} u^{\beta} = 0$.

(c) Use the Lorentz transformation to transform the f you determined in part (a) into the frame of $\mathcal{O}'$.

(d) Use the Lorentz transformation to transform u (the charge's 4-velocity) into the components in the frame of $\mathcal{O}'$. Show that $\mathbf{f} \cdot \mathbf{u} = 0$ still holds.

(e) Use your Faraday tensor from problem 2c and your u^{α} from part (d) to directly calculate the components of f^{α} as measured by $\mathcal{O}'$. Do you get the same thing as you got in part c? Should you?