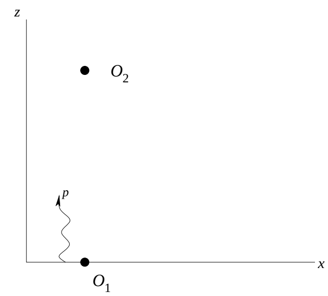
Astro 260
Group Problems #7
2005-March-18

1. Consider the two observers and the photon in the picture below. (Notice that the axes are x and z, not x and t.) Suppose these are in the Static Weak-Field Metric for a uniform gravitational field ($\Phi = g\,z$).



(a) What are the Killing vectors for this metric?

(b) Come up with a *vector* expression for the energy of the photon as measured by observer $\mathcal{O}_1$. (The easiest way to do this is to consider the problem in $\mathcal{O}_1$'s local intertial frame (the "MCRF" or Momentarily Co-moving Reference Frame). What are the coordinates of the photon's momentum $\mathbf{p}$ and $\mathcal{O}_1$'s 4-velocity $\mathbf{u}$ in this frame?)

Because the expression you come up with is a *vector* expression, then it will be true no matter which coordinate system you use to evaluate it.

(c) In the coordinates of the static weak-field metric, what are the components of $\mathcal{O}_1$'s 4-velocity $\mathbf{u}_1$? (Consider the normalization critereon $\mathcal{O}_1 \cdot \mathcal{O}_1 = -1$.)

(d) What are the components of $\mathcal{O}_2$'s 4-velocity $\mathbf{u}_2$?

(e) What is the value of $\xi \cdot \mathbf{p}$, where ξ is the time-symmetry Killing vector and $\mathbf{p}$ is the photon's mometum 4-vector, when the photon is where it is drawn above (i.e. at the same z as $\mathcal{O}_1$)? Because of the time symmetry, this vector product should be a constant of the photon's motion.

(f) What are the components of $\mathbf{p}$ for the photon when it is at $z = h$, i.e. next to $\mathcal{O}_2$?

(g) What is the energy of the photon as measured by $\mathcal{O}_2$? (Hint: refer back to the vector equation you got in (b), and generalize it for$\mathcal{O}_2$.)

(h) Does E_1/E_2 (where E_1 is the energy as measured by $\mathcal{O}_1$when the photon is near him, and E_2 is the energy as measured by $\mathcal{O}_2$when the photon is near her) give you the result you expected from gravitational redshift?

Using Killing vectors is often a more direct way to get gravitational redshift than doing things like considering time dilation and the delay between wave crests.

. . . continued on reverse. . .

2. (*Hartle 8.4, almost*) The line element of *flat spaceitme* in a frame $(t,\ x,\ y,\ z)$ that is rotating with an angular velocity Ω about the z-axis of an inertial frame is

$$ds^2 \;=\; -[1-\Omega^2(x^2+y^2)]dt^2 \;+\; 2\Omega(ydx \;-\; xdy) \;+\; dx^2 \;+\; dy^2 \;+\; dz^2$$

(a) Verify this by transforming to polar coordinates and checking that the line element is (7.4) with the substitution $\phi \;\longrightarrow\; \phi - \Omega t$. (Equation 7.4 just gives the standard spherical polor coordinate line element in flat space. If you like, you could instead start with cylindrical coordinates.)

(b) Find the geodesic equations for x, y, and z in the rotating frame.

(c) Show that in the nonrelativisitc limit these reduce to the usual equations of Newtonian mechanics for a free particle in a rotating frame exhibiting the centrifugal force and the Coriolis force. (In case you don't have those memorized, the effective force in a rotating frame is;

$$\vec{F}_{eff} \;=\; -m\,\vec{\omega}\,\times\,(\vec{\omega}\,\times\,\vec{r}) \;-\; 2\,m\,\vec{\omega}\,\times\,\vec{v}_r$$

where $\vec{\omega}$ is $\Omega\,\mathbf{e}_z$, the vector describing the rotation of the frame.