Astro 260
Group Problems #9
2005-April-16

The Flat Robertson-Walker Metric:

$$ds^2 = -dt^2 + a^2(t)\left(dx^2 + dy^2 + dz^2\right)$$

$$ds^2 = -dt^2 + a^2(t)\left(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2\right)$$

1. In this problem, you will develop expressions for the coordinate distance and the look-back time to an object who is at comoving distance r away from us.

   (a) Consider a flat Universe which has two components in it: cold matter and vacuum energy. The density (in units of critical density) of these two components *today* are $\Omega_{\rm M}$ and Ω_Λ. (The book uses Ω_v for vacuum, but Ω_Λ is slightly more standard notation.) The critical density is

   $$\rho_c = \frac{3\,{H_0}^2}{8\pi},$$

   where H_0 is the present expansion rate of the Universe.

   Write down $\rho_{\rm M}$ and ρ_Λ (which are both potentially functions of time) in terms of $\Omega_{\rm M}$, Ω_Λ, H_0, and a (the scale factor).

   (b) Consider the Friedmann Eqation for the flat Universe:

   $$\dot{a}^2 - \frac{8\pi\,\rho}{3}a^2 = 0$$

   Rewrite the Friedmann equation as a differential equation between z and t, where z is the cosmological redshift: $(1+z) = a_0/a(z)$, where $a_0 = a(t_0)$ is the scale factor today (assume $a_0 = 1$), and $a(z)$ is the scale factor when light was emitted by an object that we observe to have redshift z.

   (c) Show that the *lookback* time to an object at redshift z (that is, the amount of Robertson-Walker time t that has elapsed in between the emission and detection of light from an object whose redshift we observe to be z) has the value:

   $$t_0 - t_{\rm emit} = \int_0^z \frac{-dz}{H_0\,(1+z)\,\left[\Omega_\Lambda + \Omega_{\rm M}(1+z)^3\right]^{1/2}}$$

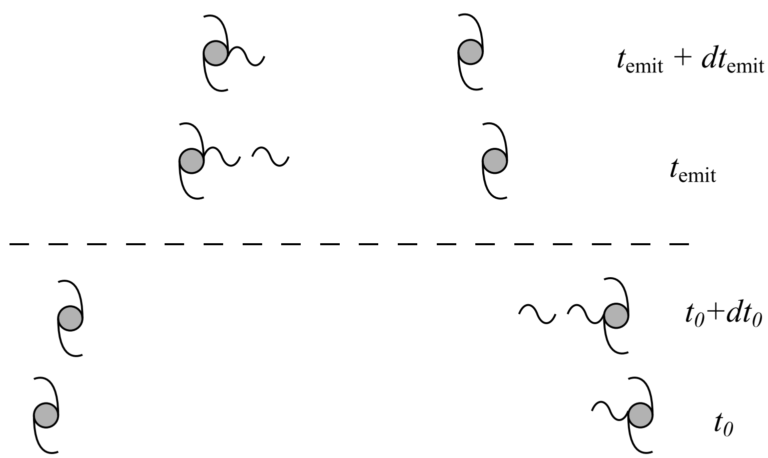
   I hope you can understand my sloppiness with z; if not, then replace z with z' everywhere inside the integral, and replace dz with dz', and then it all makes sense.

(d) Write down a similar expression for r, the *coordinate* distance to that object. (Hint: consider dr/dt for a photon travelling from there to here.)

(e) What is the *physical* distance from us to the emitting galaxy holding t constant at $t = t_0$?

(f) What is the *physical* distance from us to the emitting galaxy holding t constant at $t = t_{\rm emit}$? (Your answer should be in terms of z, H_0, $\Omega_{\rm M}$, and Ω_Λ.)

2. *Cosmological Time Dilation*: Consider the situation below:



A distant galaxy of measured redshift z emits a photon at time $t_{\rm emit} + dt_{\rm emit}$ (when it has redshift $z + dz_{\rm emit}$), and another one at time $t_{\rm emit}$ (when it has redshift z). We detect the first photon at time $t_0 + dt_0$ (note: both $dt_{\rm emit}$ and dt_0 are negative), and the second photon now (at time t_0). We can be said to have had a redshift dz_0 at the detection of the first photon, as the scale factor was *slightly* below 1 then, as shown.

(a) Write down an expression for $dz_{\rm emit}/dz_0$ in terms fo H_0, z, $\Omega_{\rm M}$, and Ω_Λ. This should *not* be an integral expression! *Hint:* both photons travel the same *coordinate* distance! You developed that expression for the *second* photon in the previous problem. *Hint 2:* Remember that:

$$\int_{x_0}^{x_0+\Delta x} f(x)dx \simeq f(x_0)\Delta x$$

for small Δx small.

(b) Develop an expression for $dt_0/dt_{\rm emit}$ in terms of H_0, z, $\Omega_{\rm M}$, and Ω_Λ. This should not be an integral expression! This problem may be a little tricky. *Hint:* If you can work out an expression in terms of dz_0 and $dz_{\rm emit}$, you can use part (a) to get rid of one of the two. You can *also* work out an expression just between dz_0 and dt_0, if you think about it.

3. *Luminosity Distance*: We define luminosity distance d_L such that:

$$F = \frac{L}{4\pi \, {d_L}^2}$$

where L is the luminosity of an emitting object, and F is the detected flux (energy per time per area). In a euclidean geometry where nothing is expanding, d_L is exactly the same as regular distance.

(a) If L is the luminosity of the emitting object, what is dN/dt', the number of photons emitted by that object in its own proper time dt'? (For simplicitly, assume all the photons are of the same frequency.)

(b) If the object is at *coordinate* distance r, what is A, the area of the sphere over which the photons emitted at the same time as the ones we detect from the object are spread?

(c) Write down an expression for $\frac{dN}{dA\,dt}$, where dN is the number of photons emitted by a distance source in its proper time dt' and dt is the amount of time it takes *us* to detect those photons. Your expression should be in terms of L, ω' (the frequency of the emitted photons), and r.

(d) Write down an expression for $\frac{dE}{dA\,dt}$, the flux we detect, in terms of the quantities from (c) and ω, the detected photon frequency.

(e) Convert your expression in (d) to an expression in terms of z, Ω_{M}, Ω_Λ, and H_0. This is the standard *luminosity distance* formula. By measuring F and z for a lot of objects of known L, we can fit the values of the parameters H_0, Ω_{M}, and Ω_Λ to this curve.