A260 Homework #4
Due Friday, 2005-Apr-1 (yes, really!)
(at the beginning of class)
(will not be extended!)

Do not use Mathematica or similar for this problem set! The algebra is not really that bad, and there isn't any calculus to speak of. It's good to get out and exercise those leg muscles every so often. The first problem will produce results that you will need for subsequent problems. Be sure to talk to others in the class about the first problem to make sure that you understand what's going on before diving into the solo problems.

1. Consider an observer $\mathcal{O}_1$ who is orbiting a black hole of mass M at the ISCO ($r = 6M$, where r is the usual Schwarzschild coordinate). This observer is in a circular orbit. Define the constant:

$$\Omega = \frac{d\phi}{dt}$$

(Note that that is a derivative with respect to the Schwarzschild t coordinate, *not* with respect to $\mathcal{O}_1$'s proper time!)

   (a) What is Ω in terms of M? (Hint: recall the definition of l, and recall that you know what l is for the ISCO. Determine Ω explicitly from this knowledge rather than simply plugging into equation 9.46 in the book.)

   (b) What are the components (in Schwarzschild coordinates) of $\mathcal{O}_1$'s 4-velocity u_1^α?

   (c) We have seen that the Schwarzschild t coordinate is also time for a very distant observer. $\mathcal{O}_2$ is a very distant observer. If $\mathcal{O}_2$ watches $\mathcal{O}_1$make one complete orbit around the black hole, how much time will elapse on $\mathcal{O}_2$'s watch?

   (d) How much time elapses on $\mathcal{O}_1$'s watch as $\mathcal{O}_1$makes one complete orbit around the black hole?

2. [***Solo Problem***] $\mathcal{O}_1$ emits a photon of frequency ω_1. This photon is moving radially (so that $p^\phi = 0$ and $p^\theta = 0$ for the photon everywhere along its path).

   (a) What are the components of the photon's 4-momentum p^α when it is emitted?

   (b) What is the value of the constant $\xi \cdot \mathbf{p}$, where ξ is the time-symmetry Killing vector?

   (c) What are the components of the photon's 4-momentum p^α when it arrives at very distant observer $\mathcal{O}_2$?

   (d) What frequency ω_2 does $\mathcal{O}_2$ measure for the photon?

   (e) In class we determined the gravitational ω_2/ω_1 for the case where $\mathcal{O}_1$ was at fixed r, θ, *and* ϕ. Is the gravitational redshift you determined in (d) a *bigger* or *smaller* effect than the case where $\mathcal{O}_1$ is not orbiting freely?

   (f) Suppose that $\mathcal{O}_1$ emits a green photon. What color (or in what part of the EM spectrum) is the photon detected by $\mathcal{O}_2$?

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3. [***Solo Problem***] Many astrophysical X-ray sources are powered by an accretion disk around a black hole, heated by gravitational potential energy released from infalling material. We will make a toy model of this process in this problem. We will start with a small object of mass m at $r \to \infty$. Mass m, having graduated from sliding down inclined planes, will now fall radially to the ISCO of a Schwarzschild black hole. Once it's there, we will stop its fall and put it into the ISCO; any excess kinetic energy left over from the fall will be converted (somehow) to a single photon, which we will then propagate back out to infinity.

   (a) When the mass m is a large distance away from the black hole $r \gg 2M$, and initially at rest, what are the components of its 4-momentum p^α?

   (b) What is the value of the constant $\xi \cdot \mathbf{p}$, where **xi** is the time-symmetry Killing vector?

   (c) What are the components of p^α when mass m reaches the ISCO, $r = 6M$?

   (d) What is the total energy E of mass m as measured by $\mathcal{O}_1$ (who is orbiting in the ISCO, as from Problem 1)?

   (e) Mass m's motion is very quickly changed to be moving along with $\mathcal{O}_1$ (i.e. it is now orbiting freely in the ISCO). After this change, what will $\mathcal{O}_1$ measure as the total energy of mass m?

   (f) What is the difference of (e) and (d)? (Do you feel like you're doing your taxes?) This is the amount of kinetic energy that was extracted from the falling mass when it was stopped and put into the circular orbit.

4. [***Solo Problem***] This problem builds on the last one.

   (a) A photon with the energy that you found in (2f) is created. This photon is sent out radially. What are p^α_{phot}, the components of the 4-momentum of this photon?

   (b) The photon goes back out to infinity. What is the energy of the photon E_2 measured by observer $\mathcal{O}_2$ (who's way out there)? This energy should be expressed as a fraction of m, the rest mass of the object dropped towards the black hole. How does this compare to the efficiency of extracting energy via nuclear fusion (where the energy extracted is typically around 1% of mc^2)?

   (c) Suppose that instead of converting the energy into a radial photon (as in (a)), that you use the energy to heat up the local gas. Using $kT \sim KE$, where KE is the excess kinetic energy of the infalling mass (i.e. your answer to (2f)), what will the temperature be of this mass? (Since we're using a thermo/stat mech result here, we're implicitly assuming that we're repeating this with lots of little mass m's to build up a gas at the ISCO.)

   (d) Assume that mass m is a hydrogen atom. Numerically evaluate T from (c). What will the peak wavelength of the blackbody spectrum emitted at this temperature be?

   (e) The photons from (d) propagate back out to infinity to be observed by $\mathcal{O}_2$ (who is an astronomer with telescopes). What *observed* temperature T would $\mathcal{O}_2$ see for the light emitted by the blackbody spectrum discussed in (d)?