Astro 260
Group Problems
2007-January-23

1. [*Hartle problem 2.5*] Calculate the area of a circle of radius r (distance from center to circumference) in the two-dimensional geometry that is a surface of a sphere of radius a. Show that this reduces to $\pi\, r^2$ when $r \ll a$.

In general, is $A < \pi\, r^2$, or is $A > \pi\, r^2$, or does the direction of the inequality depend on r? Answer this from your algebraic answer, and then answer it from just thinking about the geometry. Do you get the same answer?

2. **(a)** Show that the line element for the two-dimensional geometry that is a surface of a sphere can also be written as:

$$ds^2 \;=\; a^2 \,\frac{d\xi^2}{1-\xi^2} \;+\; a^2\,\xi^2\,d\phi^2$$

with an appropriate variable substitution.

(b) Consider the circle described by $\xi = \xi_0$. Use the line element above to calculate R, the physical radius (within this two-dimensional space) of the circle, and C, the physical circumference of this circle.

(c) From the ratio of C/R that you get in part (b), can you say that $C > 2\pi\, R$, $C < 2\pi\, R$. . . or does it depend on the value of R? Does your answer make sense if you think about the geometry from the point of view of a hyperdimensional 3d creature such as yourself? How about in comparison to the first problem?

3. Consider the geometry described by the line element:

$$ds^2 \;=\; a^2 \,\frac{d\xi^2}{1+\xi^2} \;+\; a^2\,\xi^2\,d\phi^2$$

(So far as I know, it is not possible to represent this geometry as a surface in three-dimensional space, as we did with the geometry in problem 2. This line element, however, does describe a reasonable curved two-dimensional geometry.)

(a) Calculate C/R (in terms of R/a) for a circle in this gemetry.

(b) Does this geometray have $C \simeq 2\pi\, R$ for $r \ll a$?

(c) Is $C > 2\pi\, R$, $C < 2\pi\, R$, or dies the direction of th einequality depend on R?

4 [*This one is harder, and requires you to have some familiarity with Phys. 227, and to have understood the Lagrangian stuff from Chapter 3. Don't do it in class unless you're way ahead. I will post solutions later.*]

Consider the geometry described by the following line element:

$$ds^2 = \left(1 - \frac{2\,\Phi(\vec{r})}{c^2}\right)\,(dr^2 + r^2\,d\phi^2)$$

where c is the speed of light and Φ is the Newtonian gravitational potential. Use the potential for our solar system:

$$\Phi(\vec{r}) = \frac{-G\,M_\odot}{r}$$

where $G = 6.67 \times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}$ and $M_\odot = 2.00 \times 10^{30}\,\mathrm{kg}$.

(a) Write down the Lagrangian for a test particle moving in this geometry under this potential.

(b) Derive the r and ϕ equations of motion from this Lagrangian.

(c) Explicitly identify how these equations differ from the equations you get in flat space ($ds^2 = dr^2 + r^2 d\phi^2$).

(d) For the Earth (orbiting at a distance of $1\,\mathrm{AU} = 1.496 \times 10^{11}\,\mathrm{M}$), by how much do these equations differ from the flat-space equations? (I.e. are they of the same order, are the additional terms smaller by 10^{-3}, etc.?)

(e) What condition on the radius of the orbit r would need to be satisfied for these terms to start to be significant?

(f) Compare the solutions you get for something in an elliptical orbit starting with ($r = 1\,\mathrm{AU}$, $\dot{r} = 0$, $\phi = 0$, and $\dot{\phi} = 5\,\mathrm{rad/year}$) for the flat and curved spaced values. (Obviously, doing this numerically requires a computer, and I don't really expect anybody to get this far in class.)

(g) Compare the solutions you get for something in an elliptical orbit starting with ($r = 10^{-6}\,\mathrm{AU}$, $\dot{r} = 0$, $\phi = 0$, and $\dot{\phi} = 5 \times 10^9\,\mathrm{rad/year}$) for the flat and curved spaced equations. Qualitatively how could you describe the orbit in the curved space in comparison to the ellipse you get in flat space?

This geometry is the *spatial* component of the line element you get in the Newtonian limit (the not-within-your-answer-to-part-e limit) of GR. However, it's not entirely physcially reasonable, because we've left out a term on the time component of the spacetime line element, and have assumed standard absolute time.