Astro 260 Group Problems
2007-February-08

1. Consider the static weak-field metric for a spherically symmetric gravitational field arising from a body of mass $M_\oplus = 6.0 \times 10^{24}\,\mathrm{kg}$ (i.e., the Earth). Suppose that somebody is at constant radius $r = R$ but is orbiting with constant angular velocity ω as measured by a very distant observer. (For simplicity, at some point you will probably want to set $\theta = \pi/2$. The motion will be in a plane, and since we have spherical symmetry, you have the freedom to choose θ wherever it is.)

   Just in case you care, $G = 6.67 \times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}$ and $c = 3.00 \times 10^8\,\mathrm{m\,s^{-1}}$.

   (a) If you want to go into the frame of reference of the somebody, using coordinates (t', r', θ', ϕ'), what would be a simple coordinate transformation you could perform? (HINT: this is easy; the coordinate transformation is very straightforward and what you would expect even without relatifity. The coordinates you are looking for *aren't* necessarily the "proper" coordinates of the somebody, but things that will make calculations possible. Consider that we used the static weak-field metric to do calculations of time dilation on the surface and in orbit of Earth, even though the coordinates in that metic weren't proper time/length at either of those two locations.)

   (b) Write down the line element for the somebody in terms of t', r', θ', ϕ'.

   (c) To stay in the static weak-field limit, not only do we need $2GM/r \ll c^2$, but we also need $wr \ll c$. Given this, convert your answer to b so that it is only to "first order in $1/c^2$, i.e. so that it doesn't have any (dinky)^2 terms in it. (You may notice that there is a (dinky)$^{1/2}$ term in your expansion...but still keep the first-order (dinky) terms.)

   (d) For what observer are the coordinates $(t', r', \theta', \mathrm{and}\phi')$ coordinates of proper time and proper distance?

   (e) What is the time dilation factor $d\tau/dt'$ if the somebody is standing on the equator of the Earth ($R_\oplus = 6,400\,\mathrm{km}$)? (You should be able to figure out ω from numbers you just know....) Write your answer in the form $1 \pm \epsilon$, because your answer will be 1 to any reasonable number of significant figures. Which effect dominates, gravity or special relativity?

   (f) What is the time dilation factor $d\tau/dt'$ if the somebody is orbiting at an alittude of 400km? (Use standard classical Newtonial gravity to figure out ω in this case.) Which effect dominates, gravity or special relativity?

   (g) What is the ratio $d\tau(\mathrm{orbit})/d\tau(\mathrm{surf})$ between the time dilation for the two observers of the previous two parts? Is it > 1 or < 1? Is this consistent with our previous conclusion that the orbiting clocks should be running slower?