Astro 260 Group Problems
2007-February-20

Schwarzschild Metric:

$$ds^2 \;=\; -\left(1-\frac{2M}{r}\right)dt^2 \;+\; \left(1-\frac{2M}{r}\right)^{-1}dr^2 \;+\; r^2(d\theta^2 \;+\; \sin^2\theta\,d\phi^2)$$

1. In the past, we've looked at some curved spaces and thought about how they are curved by considering the radius and circumference of circles. Here, with the Schwarzschild Metric, you can't do a circle all the way around the black hole, because the metric doesn't work right at the horzion ($r = 2M$), so you won't be able to integrate along the radius line all the way to the center. To be safe with the Schwarzschild Metric, you have to stay at $r > 2M$.

As such, instead, consider the following: start at some coordinate $r = r_1$, with $r_1 > 2M$. Make a circle staying at $r = r_1$; measure how far you've gone in space around that circle (i.e. integrate ds around holding t constant) and call that value C_1.

Next, move from r_1 to r_2, calling the total distance you go $R_2 - R_1$. (Note: this won't be the same as $r_2 - r_1$!)

Make another circle, now with $r = r_2$, and call the circumference you measure C_2.

(a) Draw a picture to make sure we all agree upon what we're doing here.

(b) If you had done this in pure flat Euclidean space, what is the value of $(C_2 - C_1)/(R_2 - R_1)$? (Your answer should just be a number!)

(c) What value do you get for $(C_2 - C_1)/(R_2 - R_1)$ around the center point of the Schwarzschild Metric? Is this value greater than, less than, or equal to the value from (b)?

(d) Recalling what you measure for C/R when your space is the *surface* of a two-dimensional sphere, and that that is positive curvature, would you call the curvature you've measured in (c) positive or negative?

...continued on reverse...

2. All answers in this problem should be in terms of m and M, and no other quantities. (Both those quantities can be taken as given.)

(a) Start with an object of mass m at rest at $r = \infty$. What energy E_1 does an observer next to this object measure for this object? (Hint: the answer is obvious, but also remember that $E = -\mathbf{u} \cdot \mathbf{p}$ where $\mathbf{u}$ is the 4-velocity of the observer, and $\mathbf{p}$ is the 4-momentum of the object observed to have energy E by the observer.)

(b) Using ξ as the timelike Killing vector of the Schwarzschild Metric, what is $\xi \cdot \mathbf{p}$ for this object?

(c) Now let the object fall to the outside of a black hole of mass M, continuing to fall freely to $r = 6M$. (Does this radius have any particular meaning?) An observer is expending a lot of rocket fuel to remain at rest at this location (i.e. with $dr/d\tau = 0$, $d\theta/d\tau = 0$, and $d\phi/d\tau = 0$). What are the components of the 4-velocity $\mathbf{u}$ for this fuel-wasting observer? (Hint: the answer here is not entirely obvious.)

(d) What is the value of the 0th component of the 4-momentum $\mathbf{p}$ for object for the falling object as it reaches $r = 6M$? (Hint: use the Killing vector.)

(e) What energy does the fuel-wasting observer measure for this falling object?

(f) The fuel-wasting observer takes the falling object and converts the *difference* between its total energy and m into a photon. What is the frequency ω of this photon?

(g) The photon propogates back out to infinity (or, at least, really far away). What frequency does an observer at rest with respect to the Schwarzschild coordinates measure for the photon?

(h) What is the ratio $E_\gamma/(mc^2)$, where E_γ is the energy of the photon measured by the observer at rest far from the black hole, and m is the mass of the object originally dropped into the black hole? (If you think about this, you should realize that your answer ought not depend on m....)

This problem is relevant to the fuelling of Active Galactic Nuclei in a way that will be discussed in class in the future.