Astro 260, Spring 2007
Homework Set 5
Due to my office the afternoon of Friday, 04/06

1. You are bold. You are very bold. You wish to hover just over the event horizon of a black hole. You are so bold that you have received the unlimited government funding necessary to produce rockets with enough thrust to hover you over the event horizon of a black hole for a while, and to escape afterwards. Unfortunately, there is one condition on the funding. You must survive the trip. Work, work, work.

This means you need to choose your black hole wisely. See, the curvature of space near the event horizons of some black holes is so huge that the tidal forces (difference in acceleration from one side of an object to another) can tear a normal person apart. Some black holes are safer than others, however. In this problem, you will figure out how to choose your black hole.

Consider only Schwarzschild black holes. It will be of use to you to know that on page 546 of Hartle, you can find the metric and all of the Christoffel Symbols for this geometry.

For this problem, define the following variables. h is the coordinate distance you will hover above the event horizon. We will assume that $h \ll 2M$, because you want to get close. (This should also be your cue to make use of the binomial theorem!) We'll assume you're hovering head-up and feet-down. Δr will be the difference between the r-coordinate of your feet and of your head. l is your height. M is the mass of the black hole. Until we get numerical, express your answer in terms of these quantities, except where noted.

This problem talks about 4-acceleration. Although we haven't talked about it in class, you can find in Hartle that its definition is:

$$a^\alpha = \frac{d^2 x^\alpha}{d\tau^2} = \frac{du^\alpha}{d\tau}$$

(a) When you are hovering, what are the components (in Schwarzschild coordinates) of your 4-velocity u^α?

(b) Were you fall freely, you would move according to the geodesic equation:

$$\frac{du^\alpha}{d\tau} = -\Gamma^\alpha_{\beta\gamma}\, u^\beta\, u^\gamma$$

Calculate the components of your 4-acceleration a^α, again in Schwarzschild coordinates, were you to fall freely. Take advantage of the binomial theorem to simply this.

(c) Consider the r-component of your 4-acceleration. What is the *difference* in this component between your feet and your head? Express your answer in terms of M, h, and Δr. (Use the binomial theorem to simply this expression, knowing that $\Delta r \ll 2M$ and $h \ll 2M$; when you do this, h should go away! Your answer will depend only on M and Δr.)

(d) Of course, Δr isn't terribly useful, since it's just a couple of coordinates and we don't really know what it is at the moment. Write down an expression for Δr in terms of l and the other quantities of this problem. And, hey, be *really* bold...let your feet touch that event horizon. Set $h = 0$. (You will find that the integral becomes far more tractable if you make the substitution $r' = r - 2M$ and recognize that everywhere in the range of integration $r' \ll 2M$.)

(e) You're bold, but you can only take so much tearing. You want the difference in the accelerations between your feet and your head to be $< 10g$, that is, less than $100\,\mathrm{m\,s^{-2}}$. (Make sure you properly convert this to geometrized units!) You are very tall, so use $l = 2m$ for your height.

What are the masses of the black holes you can safely hover over? Express your answer in terms of solar masses. Be very careful with units! You may want to convert all of your expressions for length and acceleration and so forth into physical units with the right number of factors of G and c. Where might you find such a black hole?

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2. Pick a comfortable black hole from the previous problem— one where the tidal acceleration between your foot and your head is just $1g$. Go hover over it.

Are you back already? You can't be, because there's time dilation. So calculate that instead of really going. (This isn't a lab class.)

- **(a)** If you hover with your head at $l = 2\text{m}$ of physical distance over the event horizon of the black hole for 1 year (as measured on your watch— which you are wearing on your forhead, for no adequately explained reason), how much time passes in the outside world? (Don't worry about differential time dilation between your head and your feet.)
- **(b)** Suppose that you want to achieve the same feat of time dilation, but rather through special relativity. Assume you're off in flat space and moving at constant velocity. How fast would you need to go in order to have the same amount of time pass in the outside world as you found in part (a) while one year elapses for you? (For your answer, write down both v (in speed-of-light units) and the corresponding gamma factor.)

3. [***Solo Problem***] In this problem, you will attempt to use spacetime diagrams to demonstrate how galaxies with which we are currently in causal contact will go *out* of causal contact if the expansion of the Universe is accelerating. You will want to use the Friedmann equation from Group Problem #3, and will want to consider the Flat Robertson-Walker metric:

$$ds^2 \;=\; -dt^2 \;+\; a^2(t)\,(dx^2 \;+ dy^2 \;+ dz^2)$$

with $a(t_0) = 1$. (t_0 is right now.)

- **(a)** Draw a spacetime diagram where the vertical axis is t and the horizontal axis is the *physical* distance $a(t)x$ from us to other galaxies which are off in the x-direction. Put us at $t = t_0, x = 0$. Assume that $\dot{a} > 0$, but for now don't worry about $\ddot{a}$. Draw our world line (well, OK, that's the t-axis, I'll give that away), and the world line for another galaxy which is at co-moving coordinates $x = x_g$, $y = 0$, and $z = 0$.
- **(b)** Draw a second spacetime diagram where the vertical axis is t and the horizontal axis is the comoving coordinate x. Draw the worldlines for us and for the galaxy at $x = x_g$.
- **(c)** Draw our future lightcones on your diagram from (b) assuming that we live in a vacuum-energy dominated universe ($\rho_{\text{vac}} = \rho_{\text{critical}}$). ($w = -1$ for vacuum energy.) What is the largest co-moving coordinate for a galaxy which is in our future lightcone?

  Hint: remember that a light cone is the path that a photon will follow, and that light always follows a path such that $ds^2 = 0$. Recall that in class we found $a = a_0\, e^{-H_0 t}$ for a vacuum-dominated Universe.

  Another Hint: figure out an expression $x(t)$ for light rays that cross the event of $t = t_0, x = 0$ (which is us right now).
- **(d)** Draw our *past* lightcones on your diagram from (b), making the same assumptions as you did in (c).
- **(e)** Now consider the event that is us at some time $t = t_1$ in the future. What is the largest co-moving coordinate that is now in our future lightcone? Is this larger or smaller than the value from (c)? Draw this point and these future lightcones on your diagram from (b). Show that you can pick an x_g such that a distant galaxy is in our future lightcone now ($t = t_0$), but is *no longer* in our future lightcone in the future (at $t = t_1$).

  That galaxy used to be in causal contact with us (it could get signals at c from us), but is no longer. (Sad.)

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4. This problem is all about dropping things from far away into black holes.

(a) If you start with an object at rest and very far from a black hole, its 4-velocity is $(u^0 = 1, u^i = 0)$, because Kerr metric approaches the SR metric for very large r.

If the black hole is a *Schwarzschild* black hole, make an argument to convince me that the object will be moving only *radially* as it approaches the black hole's event horizon. (There are a few ways to do this.)

(b) Now consider an extremal Kerr black hole (i.e. $J = M^2$, and thus $a = M$). As discussed in class and in Hartle, the Kerr metric has two killing vectors, $\xi = (1, 0, 0, 0)$ and $\eta = (0, 0, 0, 1)$. What are the values of $\xi \cdot \mathbf{u}$ and $\eta \cdot \mathbf{u}$ for the trajectory of something that starts very far away from the black hole?

(c) Assume that the particle is falling in along the equator of the black hole, i.e. at $\theta = \pi/2$. Although you don't have to write about it, make sure you understand the symmetry arguments that argue that it will *stay* at $\theta = \pi/2$, and henceforth use the simplified metric for that value of θ.

What is the value of u^α when the object just reaches the outside of the Ergosphere, i.e. $r = 2M$? Although it may not be obvious, you will need to use *both* killing vectors *and* the normalization of 4=velocity to solve this. Don't forget the cross-term (the $dtd\phi$ term) in the Kerr metric! It's important here!

(d) What does your answer to (c) tell you about the particle's trajectory as it falls towards Kerr black hole from rest a very large distance away?

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